

# Viscosity of net-baryon fluid near the QCD critical point

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## Abstract

In the dynamics of the QCD critical point, the net-baryon fluid, linked to the slow component of the order parameter, relaxes to a 3d Ising system in equilibrium. The transport coefficients develop power-law singularities in the limit  $T \rightarrow T_c$ ,  $\mu_b = \mu_c$ , associated with the critical exponents of the 3d Ising universality class. An analytical study of shear and bulk viscosity, with constraints imposed by universality and the requirements of a class of strong coupling theories, is performed in the neighbourhood of the critical point. It is found that the shear viscosity of the net-baryon fluid is restricted in the domain  $1.6 \leq 4\pi_s^{\eta} \leq 3.7$  for  $T_c < T \leq 2T_c$  whereas the bulk viscosity is small,  $4\pi_s^{\zeta} < 0.05$  (for  $T > 1.23T_c$ ) but rising towards the singularity at  $T = T_c$ .

Keywords: QCD critical point, viscosity, baryonic fluid

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The baryon-number fluid, created in high-energy nuclear collisions is a distinct system, associated with the dynamics of the QCD critical point. The baryon-number density  $n_b$ , being a conserved quantity, is a slow component of the order parameter, compatible with the long-time scale of the QCD critical phenomena [1, 2]. In contrast, the chiral condensate, the other component of the order parameter, is linked to a fast mode owing to the fact that the  $\sigma$ -field becomes massive and therefore does not contribute to the dynamics of the QCD critical point [1]. As a result, one may claim that although in the study of static properties of the QCD critical point (critical fluctuations, divergence of baryon-number susceptibility) both components of the order parameter ( $\sigma, n_b$ ) are relevant [3], in any attempt to investigate the dynamic properties of the QCD critical point (divergence of transport coefficients) only the baryon-number density  $n_b$  is a relevant order parameter [1]. Experimentally, one may state that in the search for the location of the critical point in the phase diagram, the investigation of critical fluctuations of  $\pi^+\pi^-$  pairs may simulate  $\sigma$ -field critical fluctuations [4]. However, dynamic properties of the QCD critical point cannot be revealed within a study of transport coefficients (viscosity) of a meson gas produced in high-energy nuclear collisions [5].

In what follows, we consider the behaviour of viscosity (shear and bulk) in the baryon-number fluid, near the QCD critical point in a process of relaxation towards a state of equilibrium described by 3d Ising universality class. In the same universality class belong also conventional systems (He, N<sub>2</sub>, H<sub>2</sub>O) of liquid-gas transition [6] and the basic ingredients in this process out of equilibrium are thermal diffusion and sound waves [7]. The Ising model description in equilibrium is characterized by the critical exponents of the divergent thermodynamic quantities and of the correlation length, near the critical point [8]. On the basis of dimensional considerations: [viscosity]=[energy density] $\times$ [time], one may obtain the following expressions for shear ( $\eta$ ) and bulk ( $\zeta$ ) viscosity in terms of singular quantities in the limit  $T \rightarrow T_c$ ,  $\mu_b = \mu_c$ :

$$\frac{\eta}{s} = \frac{k_B T v_s^{-1}}{\xi^2 s} F^{(s)} \left( \frac{c_P}{c_V} \right) \quad ; \quad \frac{\zeta}{s} = \frac{\rho v_s \xi}{s} F^{(b)} \left( \frac{c_P}{c_V} \right) \quad (1)$$

where  $\xi$  is the correlation length,  $v_s$  the velocity of sound waves,  $\rho$  the mass density of the medium (net-baryon fluid) in the bulk,  $s$  the entropy density and  $(c_V, c_P)$  the specific heat coefficients. The ratios (1) are dimensionless in the system of units:  $k_B = c = \hbar = 1$ . The

basic thermodynamics of the fluid is formulated in terms of the relations:

$$c_P - c_V = T k_T \left( \frac{\partial P}{\partial T} \right)_V^2, \quad \frac{c_P}{c_V} = \frac{k_T}{k_S}, \quad v_s^2 = (\rho k_S)^{-1},$$

$$s = \frac{\varepsilon + P}{T} - \frac{\mu_b n_b}{T} \quad (2)$$

where  $k_T$ ,  $k_S$  are the isothermal and adiabatic (isoentropic) compressibility,  $\varepsilon$  the energy density,  $P$  the pressure and  $\mu_b$  the baryochemical potential.

A minimal requirement of relativistic thermodynamics leads to the identification of the mass density  $\rho$  of the fluid with the enthalpy density:  $h = \varepsilon + P$  ( $\rho = \frac{h}{c^2}$ ) in eqs. (1) and (2) as a result of the properties of the energy-momentum tensor [9]. Moreover, in order to fix the amplitudes (scales) of the singular quantities, consistently with relativity, one may consider in the quark phase ( $T \gg T_c$ ) the equation of state of an ideal, massless, classical system leading to:

$$\varepsilon = 3P, \quad P = n_b T, \quad h = 4n_b T, \quad c_V = 3n_b, \quad c_P = 4n_b,$$

$$k_T = (n_b T)^{-1}, \quad s = \left( 4 - \frac{\mu_b}{T} \right) n_b \quad (3)$$

Introducing now the appropriate critical exponents and the corresponding amplitudes, we obtain in the limit  $T \rightarrow T_c$ ,  $\mu_b = \mu_c$ , the power laws:

$$c_V = A_{\pm} |t|^{-\alpha}, \quad k_T = \Gamma_{\pm} |t|^{-\gamma}, \quad \xi = \xi_{\pm} |t|^{-\nu} \quad \left( t \equiv \frac{T - T_c}{T_c} \right) \quad (4)$$

where the indices ( $\pm$ ) in the amplitudes correspond to the limits  $t \rightarrow 0^+$  and  $t \rightarrow 0^-$  respectively [8].

In eqs. (4) not only the critical exponents are universal but also the ratios of the amplitudes  $\frac{A_+}{A_-}$ ,  $\frac{\Gamma_+}{\Gamma_-}$ ,  $\frac{\xi_+}{\xi_-}$ , corresponding to the phases  $T > T_c$  (net-baryon, quark-matter fluid) and  $T < T_c$  (net-baryon, baryonic fluid), are fixed within the universality class of the critical point [8]. Moreover, following our discussion above, the amplitudes ( $A_+$ ,  $\Gamma_+$ ) in the quark-matter phase, representing the scales of the thermodynamic quantities ( $c_V$ ,  $k_T$ ) near the critical temperature, can be fixed with the help of eqs. (3) assuming a continuous transition to the ideal behaviour. To ensure the continuity of the sound velocity (see eqs. (7) below), reaching the constant value  $\frac{1}{\sqrt{3}}$  in the ideal regime, the matching of  $c_V$ ,  $k_T$  has to be taken at  $t = 1$  ( $T = 2T_c$ ) leading to the relations:

$$A_+ = 3n_c, \quad \Gamma_+ = (2n_c T_c)^{-1} \quad (5)$$

where  $n_c \equiv n_b(T_c) = n_b(2T_c)$  denotes the critical baryon-number density. Here we have made use of the fact that the order parameter  $n_b(T) - n_c$  vanishes in the symmetric phase  $T > T_c$  [11].

In this framework one may proceed to a semi-quantitative treatment of shear and bulk viscosity, near the QCD critical point, on the basis of eqs. (1, 2, 4). For the functions  $F^{(i)}\left(\frac{c_P}{c_V}\right)$ ,  $i : (s, b)$  in eqs. (1) we adopt a simple model inspired by a perturbative treatment of conventional fluids, in the vicinity of liquid-gas critical point [7]:  $F^{(i)}\left(\frac{c_P}{c_V}\right) = f^{(i)}\frac{c_P}{c_V}$  where the dimensionless constants  $f^{(i)}$  are not universal, they depend on the nature and the length scale of the medium at microscopic level. With this choice, equations (1, 2) give:

$$\begin{aligned} v_s^2 &= \frac{k_T^{-1}}{h} \left[ 1 + T k_T \left( \frac{\partial P}{\partial T} \right)_V c_V^{-1} \right] \\ \frac{\eta}{s} &= f^{(s)} \frac{T^{3/2} h^{1/2}}{s} \left( \frac{\partial P}{\partial T} \right)_V k_T \xi^{-2} c_V^{-1/2} \left[ 1 + T^{-1} \left( \frac{\partial P}{\partial T} \right)_V \frac{c_V}{k_T} \right]^{1/2} \\ \frac{\zeta}{s} &= f^{(b)} \frac{T^{3/2} h^{1/2}}{s} \left( \frac{\partial P}{\partial T} \right)_V^3 k_T \xi c_V^{-3/2} \left[ 1 + T^{-1} \left( \frac{\partial P}{\partial T} \right)_V \frac{c_V}{k_T} \right]^{3/2} \end{aligned} \quad (6)$$

Incorporating the power laws (4) for  $c_V$ ,  $k_T$  and  $\xi$  in eqs. (6) we obtain the singular forms in the limit  $T \rightarrow T_c$ :

$$\begin{aligned} v_s^2 &= \frac{|t|^\alpha}{h} \left[ \Gamma_\pm^{-1} |t|^{\gamma-\alpha} + A_\pm^{-1} T \left( \frac{\partial P}{\partial T} \right)_V^2 \right] \\ \left( \frac{\eta}{s} \right)_\pm &= f^{(s)} \frac{T_c^{3/2} h_c^{1/2} \lambda_c}{s_c} \left( \Gamma_\pm \xi_\pm^{-2} A_\pm^{-1/2} \right) (1 + T_c^{-1} \lambda_c^{-2} A_\pm \Gamma_\pm^{-1} |t|^{\gamma-\alpha})^{1/2} |t|^{-\gamma+2\nu+\frac{\alpha}{2}} \\ \left( \frac{\zeta}{s} \right)_\pm &= f^{(b)} \frac{T_c^{3/2} h_c^{1/2} \lambda_c^3}{s_c} \left( \Gamma_\pm \xi_\pm A_\pm^{-3/2} \right) (1 + T_c^{-1} \lambda_c^{-2} A_\pm \Gamma_\pm^{-1} |t|^{\gamma-\alpha})^{3/2} |t|^{-\gamma-\nu+\frac{3\alpha}{2}} \end{aligned} \quad (7)$$

where  $\lambda_c \equiv \left( \frac{\partial P}{\partial T} \right)_V$  at  $T = T_c$ . The leading power laws of shear and bulk viscosity are:

$$\eta \sim |t|^{2\nu+\frac{\alpha}{2}-\gamma} \quad , \quad \zeta \sim |t|^{-\gamma-\nu+\frac{3\alpha}{2}} \quad (8)$$

where the indices of the singularities are given in terms of the Ising critical exponents:  $\alpha = 0.11_{-0.03}^{+0.01}$ ,  $\gamma = 1.24_{-0.04}^{+0.16}$ ,  $\nu = 0.63 \pm 0.04$  [8]. The behaviour (8) is universal, it is valid near the liquid-gas critical point of conventional matter [7] and also near the quark-hadron critical point of QCD matter. The exponents of power laws (8) are restricted in the domain:

$$-0.18 \leq 2\nu + \frac{\alpha}{2} - \gamma \leq 0.2 \quad ; \quad -1.95 \leq -\gamma - \nu + \frac{3\alpha}{2} \leq -1.61 \quad (9)$$

We observe that the nature of the singularity in shear viscosity is uncertain, it may be either a softly divergent power law or a cusp singularity. In contrast, the bulk viscosity develops a rather strong divergence, in the limit  $T \rightarrow T_c$ , which may lead to observable effects. We notice that the constraint (9) on the exponents is compatible with the prediction in reference [1] which is based on a RG treatment of Langevin equation, in the dynamics of the QCD critical point:  $\eta \sim |t|^{-0.03}$ ,  $\zeta \sim |t|^{-1.7}$ .

In order to remove the uncertainty in the solution (9) which is due to the ambiguity in the actual values of the Ising critical exponents, we adopt, in what follows, the above result in reference [1] corresponding to the choice:  $\alpha \simeq 0.11$ ,  $\gamma \simeq 1.27$ ,  $\nu \simeq 0.59$  in eqs. (7). The characteristic properties of viscosity near the QCD critical point, described by the solution (7), depend on a number of nonuniversal amplitudes which are fixed by the following constraints:

(a) the assumption that in the quark-matter phase ( $T \gtrsim T_c$ ) the amplitudes  $\Gamma_+$ ,  $A_+$  are given by eqs. (5) which are compatible with the equation of state of an ideal, massless, classical system for  $T \gg T_c$  and

(b) the universality constraint imposed on the ratios of the Ising amplitudes:  $\frac{A_+}{A_-} = 0.5 - 0.6$ ,  $\frac{\xi_+}{\xi_-} = 2$ ,  $\frac{\Gamma_+}{\Gamma_-} = 4.5 - 5.0$  [10].

In fact, the solution (7) can be written in a simplified form:

$$\begin{aligned} \left(\frac{\eta}{s}\right)_{\pm} &= f^{(s)} M_{\pm} (1 + \Lambda_{\pm} |t|^{\gamma-\alpha})^{1/2} |t|^{-\gamma+2\nu+\frac{\alpha}{2}} \\ \left(\frac{\zeta}{s}\right)_{\pm} &= f^{(b)} N_{\pm} (1 + \Lambda_{\pm} |t|^{\gamma-\alpha})^{3/2} |t|^{-\gamma-\nu+\frac{3\alpha}{2}} \end{aligned} \quad (10)$$

where:

$$\begin{aligned} M_{\pm} &\equiv \frac{T_c^{3/2} h_c^{1/2} \lambda_c}{s_c} \Gamma_{\pm} \xi_{\pm}^{-2} A_{\pm}^{-1/2} \quad , \quad N_{\pm} \equiv \frac{T_c^{3/2} h_c^{1/2} \lambda_c^3}{s_c} \Gamma_{\pm} \xi_{\pm} A_{\pm}^{-3/2} \quad \text{and} \\ \Lambda_{\pm} &\equiv T_c^{-1} \lambda_c^{-2} A_{\pm} \Gamma_{\pm}^{-1} \quad , \end{aligned} \quad (11)$$

with  $h_c = 4n_c T_c$ ,  $\lambda_c = n_c$ ,  $s_c = \left(4 - \frac{\mu_c}{T_c}\right) n_c$  assuming the (approximate) validity of eqs. (3) in the temperature range  $(T_c, 2T_c)$  for all thermodynamic quantities which do not possess divergent singularities for  $T \rightarrow T_c$ . The constraints (a) and (b) lead to the following solution for the dimensionless amplitudes (11):

$$M_+ = \frac{1}{\sqrt{3}} \frac{\xi_+^{-2} T_c}{s_c} \quad , \quad N_+ = \frac{1}{3\sqrt{3}} \frac{n_c \xi_+ T_c}{s_c} \quad , \quad \Lambda_+ = 6$$

$$M_- = \frac{4}{7\sqrt{3}} \frac{\xi_+^{-2} T_c}{s_c} \quad , \quad N_- = \frac{1}{84\sqrt{3}} \frac{n_c \xi_+ T_c}{s_c} \quad , \quad \Lambda_- = 60 \quad (12)$$

where  $\xi_+ \simeq 1$  fm a typical scale of the correlation length and a set of critical values ( $T_c$ ,  $\mu_c$ ,  $n_c$ ) can be taken from reference [11] where a study of baryon-number susceptibility near the critical point is performed with the help of NA49 data [12]:  $T_c \simeq 160$  MeV,  $\mu_c \simeq 220$  MeV,  $n_c \simeq 0.13$  fm<sup>-3</sup>.

To complete our treatment and determine the remaining constants  $f^{(s)}$ ,  $f^{(b)}$  in eqs. (10), we employ as a final guiding principle, the KSS bound [13] which is assumed to be reached by the minimum of the ratio  $\frac{\eta}{s}$ , located very close to the critical temperature, in the hadronic phase ( $t \simeq -2.7 \cdot 10^{-3}$ ) according to eqs. (10). This constraint has its origin in a class of strong coupling field theories (AdS/CFT limit) and it is widely accepted that the formation of quark matter in high-energy nuclear collisions creates an ideal environment in order to test its validity [6]. Also, in the same framework, a constraint on the bulk viscosity can be obtained if we use the parametrization,  $\frac{\zeta}{s} = \frac{1}{8\pi} (\frac{1}{3} - v_s^2)$ , introduced in reference [14] and take, for our purpose, the average in the domain  $0.5 \leq t \leq 1$ . From eqs. (7) we have for  $T_c \leq T \leq 2T_c$ :

$$v_s^2 = \frac{t^\alpha}{4} \left( \frac{2t^{\gamma-\alpha}}{1+t} + \frac{1}{3} \right) \quad ; \quad \langle v_s^2 \rangle \simeq 0.28 \quad (13)$$

Thus, we obtain the final constraints:

$$\left( \frac{\eta}{s} \right)_{\min} = \frac{1}{4\pi} \quad (t \simeq 0^-) \quad ; \quad \left\langle \left( \frac{\zeta}{s} \right)_+ \right\rangle \simeq \frac{0.028}{4\pi} \quad (14)$$

which lead to the estimate:  $f^{(s)} \simeq 8.0 \times 10^{-2}$  and  $f^{(b)} \simeq 1.9 \times 10^{-3}$ .

In Fig. 1 our solution for the shear viscosity of net-baryon matter (blue line) is compared with other findings. In the hadronic phase ( $T < T_c$ ) we found  $1 \leq 4\pi \frac{\eta}{s} \leq 4.3$  for  $\frac{T_c}{2} \leq T < T_c$ , deviating from the behaviour of chiral matter (meson gas in chiral perturbation theory) [5] and the behaviour of  $\frac{\eta}{s}$  extracted from heavy-ion collisions at intermediate energies (HIC-IE) [15]. For  $T > T_c$  (quark matter) we found  $1.6 \leq 4\pi \frac{\eta}{s} \leq 3.7$  for  $T_c < T \leq 2T_c$  and a comparison with recent results of lattice QCD (lQCD) for the shear viscosity of gluonic matter [16] is illustrated. Also, in Fig. 1, predictions of perturbative QCD [17] and of a quasi-particle model [18] (yellow band) are presented for comparison. Finally, it is of interest to note that the weakness of the singularity, at  $T = T_c$ , manifests itself as a two-minima structure, very close to the critical temperature (Fig. 1). The absolute minimum reaches the KSS bound in the hadronic and not in the quark-matter phase. This structure

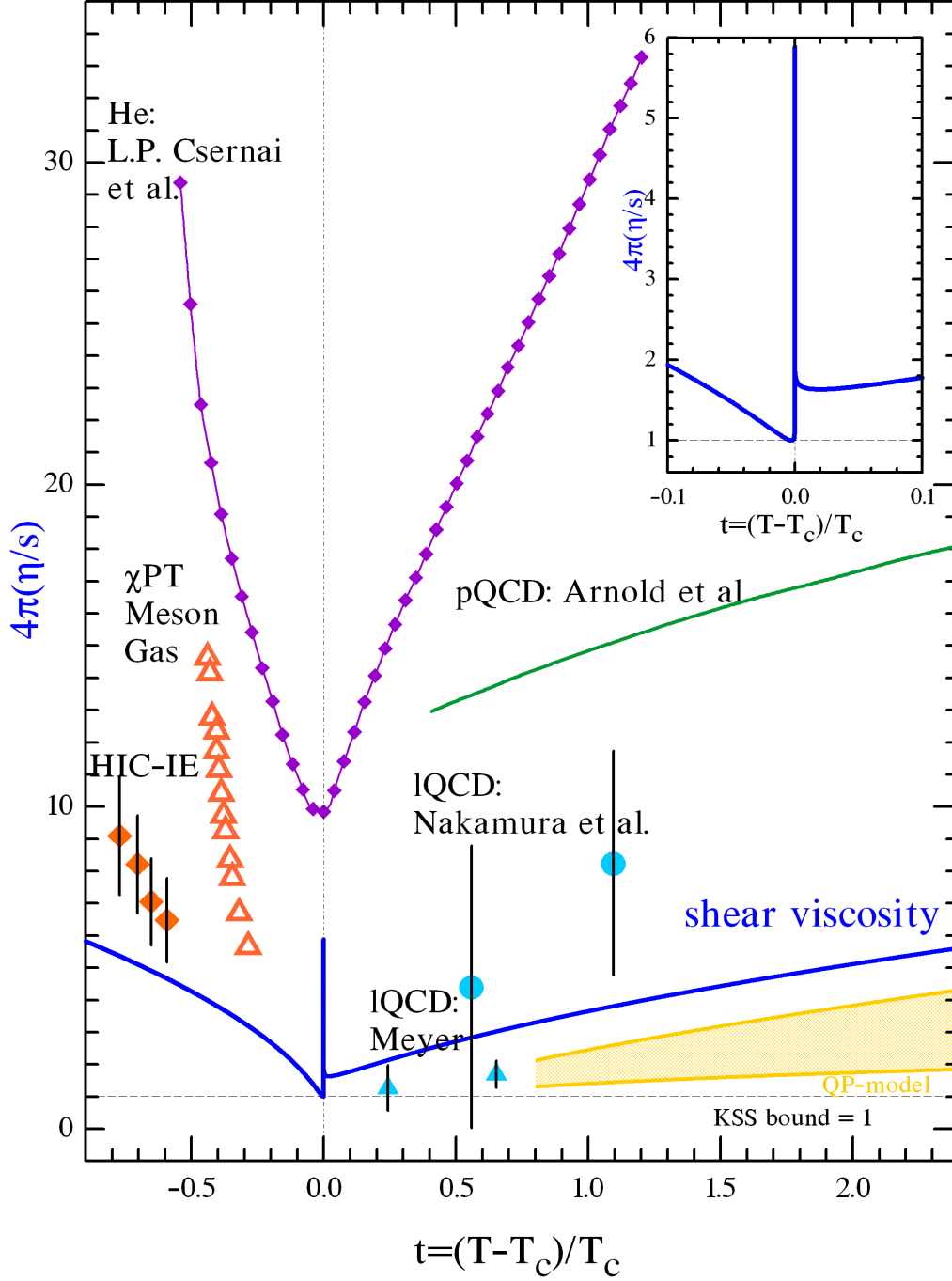


FIG. 1: Our solution for the shear viscosity (blue line) compared with the findings of [5] (empty orange triangles), [15] (solid orange rectangles), [16] (solid blue circles and triangles), [17] (green line), a quasi-particle model [18] (yellow band) and [6] (purple line and rectangles). In the embedded graph we focus on the shape of our solution in the vicinity of the critical temperature.

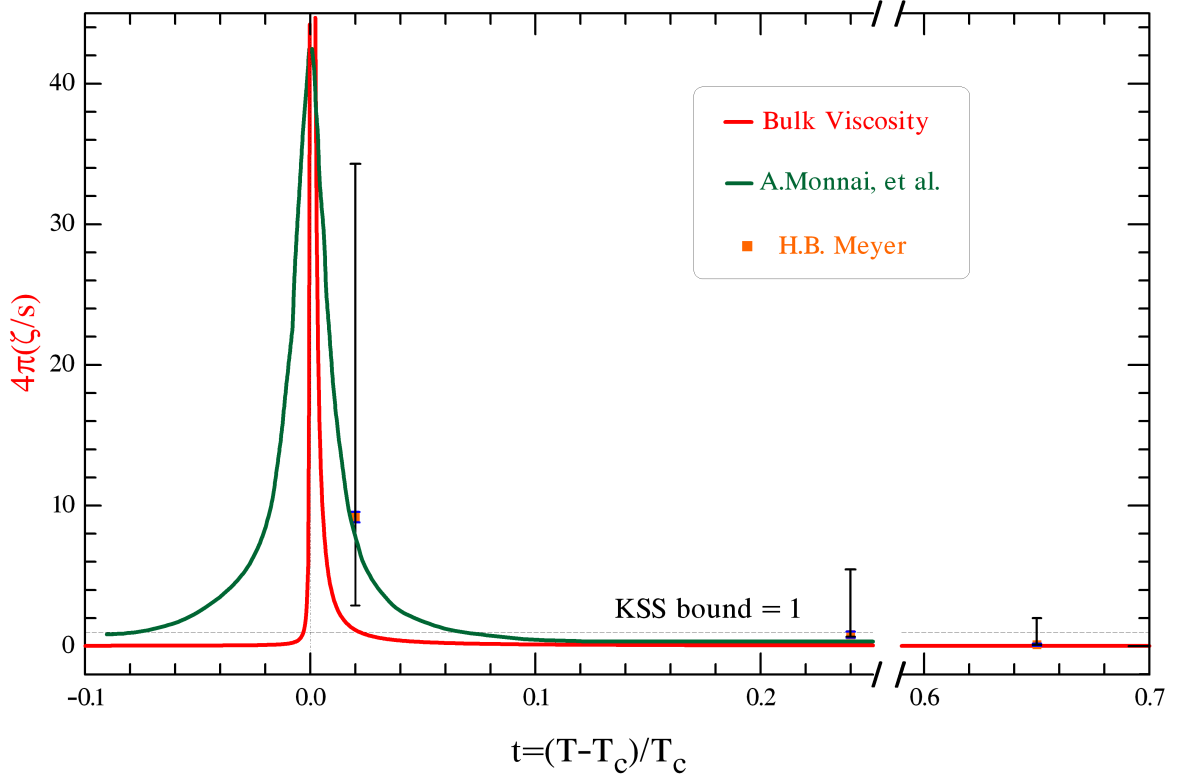


FIG. 2: Our solution for the bulk viscosity (red line) compared with the findings of [19] (green line) and [20] (solid orange rectangles) with systematic (black) and statistical (blue) uncertainties.

cannot be seen in the coarse data of conventional matter, as shown in Fig. 1 in the case of helium [6] and, certainly, is not expected to be observable in high-energy nuclear collisions either.

In Fig. 2 the bulk viscosity of net-baryon matter, in our solution, is presented (red line). Despite the fact that it develops a strong singularity at the critical point ( $\zeta \sim \xi^{2.9}$ ) it decreases rapidly, for  $|t| \geq 0.02$ , to values smaller than the KSS bound. Our result is compared with the solution (green line) in reference [19] where a dynamical treatment of enhanced bulk viscosity near the critical point is performed. In the same figure, the results of lattice QCD for gluonic matter are shown [20] whereas in a similar lQCD treatment [15] the results for the bulk viscosity of gluonic matter are compatible with zero, for  $\frac{3T_c}{2} < T < 2T_c$ , and are not shown in Fig. 2).

In summary, an analytical study of shear and bulk viscosity of net-baryon matter near the QCD critical point is performed. It is based on the assumption that the net-baryon



fluid, associated with the slow order parameter (baryon-number density  $n_b$ ) of the critical phenomenon, relaxes, in a process out of equilibrium, to the Ising universality class in equilibrium. The universal indices (critical exponents, ratios of the amplitudes of critical singularities) are basic ingredients in this approach, leading to a prediction of viscosity near the critical point. This study suggests that precision measurements of elliptic flow of net protons at the SPS (NA61 experiment) or at RHIC, in the Beam Energy Scan program [21], are of particular importance since they are strongly linked to the dynamics of the QCD critical point.

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